Lecture 3

- Recap
- Greens functions in electrostatics (cont.)
  - → residue integrals
- Image charge method
- Eigenfunctions method
- Magnetostatics
- Dynamics
  - Greens functions for wave operator

Recap

$$\begin{cases} \vec{\nabla} \cdot \vec{E} = \dfrac{\rho}{\varepsilon_0} \\ \vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \dfrac{\partial \vec{E}}{\partial t} \\ \vec{\nabla} \times \vec{E} = -\dfrac{\partial \vec{B}}{\partial t} \\ \vec{\nabla} \cdot \vec{B} = 0 \end{cases}$$

$$\vec{E} = -\vec{\nabla} \cdot \varphi - \partial_t \vec{A}$$

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

$$\begin{cases} \dfrac{1}{c^2} \partial_t^2 \vec{A} - \Delta \vec{A} = \mu_0 \vec{J} \\ \dfrac{1}{c^2} \partial_t^2 \varphi - \Delta \varphi = \dfrac{\rho}{\varepsilon_0} \\ \dfrac{1}{c^2} \partial_t \varphi + \vec{\nabla} \cdot \vec{A} = 0 \quad \to \text{Gauge condition} \end{cases}$$

- Electrostatics :

$$-\Delta \varphi = \frac{\rho}{\varepsilon_0}$$

$$-\Delta_x G(x, x') = \delta^3 (x - x')$$

Derivation of the Green's function + review of contour integrals.

$$-\Delta_x G = \delta(\vec{x}-\vec{x}') \quad (\text{in } R^3, \text{ no B.C.})$$

$$k^2 G = e^{-i\vec{k}\cdot\vec{x}'}$$

$$G = \frac{1}{(2\pi)^3}\int d^3k \, \frac{e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}}{k^2} =$$

$$\oint \frac{g(z)}{z} = 2\pi i\, g(0)$$

$$\frac{1}{(2\pi)^2}\int_0^\infty dk \, d\theta \sin\theta \, e^{ik|x|\cos\theta} \to$$

$$\frac{1}{(2\pi)^2}\int_0^\infty dk \, \frac{e^{ikx}-e^{-ikx}}{i|X||k|} =$$

$$= \frac{1}{2}\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} dk \, \frac{e^{ikx}-e^{-ikx}}{i\,k\,|X|} = \frac{1}{4\pi|x|}$$

- $\varphi(x)\big|_{\partial V} = g(x)$ → chose G_D: $G_D\big|_{\partial V} = 0$

- $\varphi(x) = \frac{1}{\varepsilon_0}\int_V d^3x' \, G(x',x)\,\rho(x) - \int_{\partial V} d\vec{\sigma}\, g(x')\,\vec{\nabla}_{x'} G_D(x',x)$

If, instead,

$$-\vec{n}\cdot\vec{\nabla}\varphi\Big|_{\partial V} = g(x) \to \text{choose } G_N : \vec{n}\cdot\vec{\nabla}G_N\Big|_{\partial V} = 0$$

$$\varphi(x) = \frac{1}{\varepsilon_0}\int_V d^3x'\, G(x',x)\,\rho(x') + \int_{\partial V} d\vec{\sigma}'\cdot G_N(x',x)\,\vec{\nabla}\varphi(x')$$

In what follows we will use uniqueness of the solution to N and D problems.

Suppose there are two solutions φ_1 and φ_2

$$\underset{i=1,2}{\Delta\varphi_i = -\frac{\rho}{\varepsilon_0}}, \quad \varphi_i\Big|_{\partial V} = g, \text{ or } \partial_n\varphi_i\Big|_{\partial V} = g$$

Consider $u = \varphi_1 - \varphi_2$, and integrate

$$\Delta u = 0$$

$$\int_V \vec{\nabla}(u\vec{\nabla}u) = \int_V (\vec{\nabla}u)^2 \Rightarrow \vec{\nabla}u = 0$$

$$\Downarrow$$

$$\varphi_1 - \varphi_2 = \text{const.}$$

$\parallel \to$ Gauss

$$\int_{\partial V} d\vec{\sigma}(u\vec{\nabla}u) = 0 \quad \text{(either } u \text{ or } \vec{\nabla}u \text{ vanishes)}$$

It follows that G_D and G_N are also unique.

Method of Images.

- It relies on uniqueness and linearity.
- Useful to construct Green's functions or potentials with B.C. with simple geometries.

plane: $z=0$

$$G_{N/D}(x,x') = \frac{1}{4\pi}\frac{1}{|\vec{x}-\vec{x}'|} \pm \frac{1}{4\pi}\frac{1}{|\vec{x}-\vec{x}'_R|}$$

$$\vec{x}'_R = (x, y, -z)$$

$G_{N/D}$ satisfies Neumann/Dirichlet B.C on the plane $z=0$.

It also satisfies Laplace equation inside the region V: $z>0$, but not outside! Uniqueness tells us that it is the right G.F. for $z>0$.

This method is applicable to other more complicated geometries.

Expansion in Eigenfunctions

This is another method to find G.F. for a region.

$$-\nabla_x^2 G(\vec{x},\vec{x}') = \delta^3(\vec{x}-\vec{x}')$$

Start with an Eigenvalue problem:

$$-\nabla^2 \psi_n(x) = \lambda_n \psi_n(x)$$

With Ψ_n satisfying corresponding boundary conditions. We also choose normalization:

$$\int_V d^3x\, \Psi_n^*(x)\, \Psi_m(x) = \delta_{nm}$$

Then $\delta^3(\vec{x}-\vec{x}') = \sum_n \Psi_n^*(\vec{x}')\, \Psi_n(\vec{x})$.

(completeness)

Now we can expand the equation for G.F. in the basis of Ψ_n's:

$$G(x',x) = \sum C_n(x')\, \Psi_n(x)$$ (x – variable, x' – parameter)

$$-\nabla_x^2\, C_n(x')\, \Psi_n(x) = \Psi_n^*(x')\, \Psi_n(x)$$

$$\Rightarrow C_n(x') = \frac{\Psi_n^*(x')}{\lambda_n}$$

$$G(x,x') = \sum_n \frac{1}{\lambda_n}\, \Psi_n^*(x')\, \Psi_n(x)$$

Magnetostatics

We now consider the situation
$\vec{E}=0 \quad \vec{B}=\vec{\nabla}\times\vec{A}$ with

$$\Delta\vec{A}=-\mu_0\vec{J}\ , \qquad \vec{\nabla}\cdot\vec{A}=0$$

This is consistent with $\vec{\nabla}\cdot\vec{J}=0$

For each component we have the same problem as in Electrostatics, thus we can use identical methods.

Solution of Maxwell Equations:

Dynamics

$$\Box\varphi=\frac{\rho}{\varepsilon_0} \qquad \Box\vec{A}=\mu_0\vec{J}$$

As before, equations for φ and $\vec{A}$ are similar, so we focus on

φ without loss of generality.
Same as in the static case, we will look first for the Greens function:

$$\Box_{x,t}\, G(\vec{x},t,\vec{x}',t') = \delta^3(\vec{x}-\vec{x}')\,\delta(t-t')$$

Then we can find potential by integrating over space and time:

$$\varphi(x,t) = \frac{1}{\varepsilon_0}\int d^3x'\,dt'\,G(\vec{x},t,\vec{x}',t')\cdot\rho(x',t')$$

As in the static case, GF depends on boundary conditions. We will consider dynamics in entire space ($G \xrightarrow[|\vec{x}|\to\infty]{} 0$)

However, we now need to specify the "boundary conditions" in time.

These will correspond to vanishing of the G.F. in the past (retarded) or in the future (advanced). We will use retarded most of the time.

To compute the G.F. we will again use Fourier transform

$$\Box G = \delta^3(x-x')\,\delta(t-t')$$

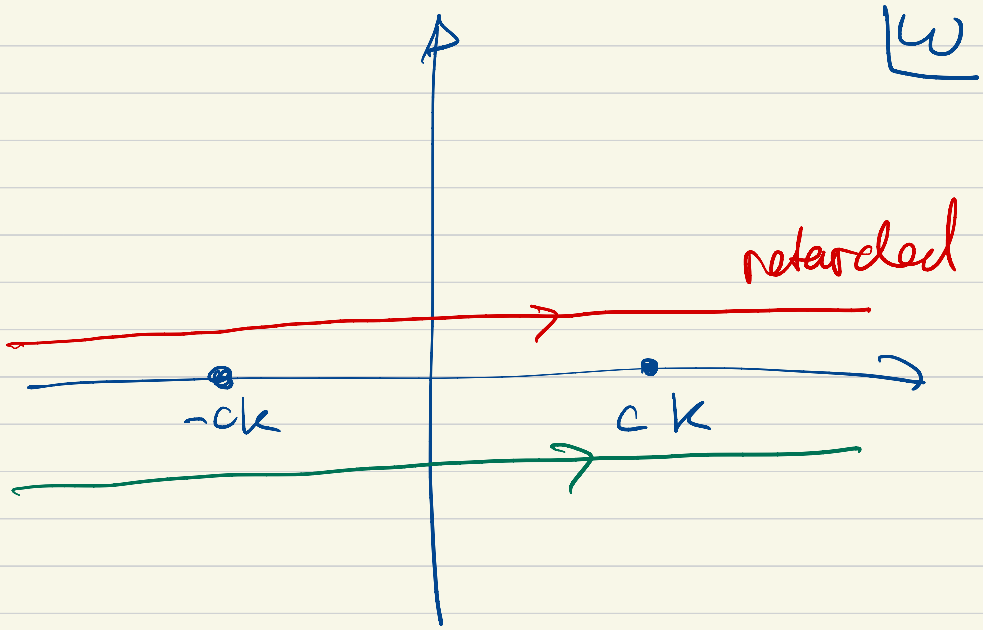
$$\Downarrow$$

$$\left[-\frac{\omega^2}{c^2}+k^2\right]\hat{G}(\omega,k) = e^{-ikx'+i\omega t}$$

$$\hat{G} = \frac{1}{-\frac{\omega^2}{c^2}+k^2}$$

$$G = \int \frac{d^3k\, d\omega}{(2\pi)^4}\, e^{ikx-i\omega t}\,\hat{G}$$

Let us first take the integral over ω. This integral is not well-defined because of the singularities at $\omega = \pm ck$. These are resolved by changing the contour of integration:

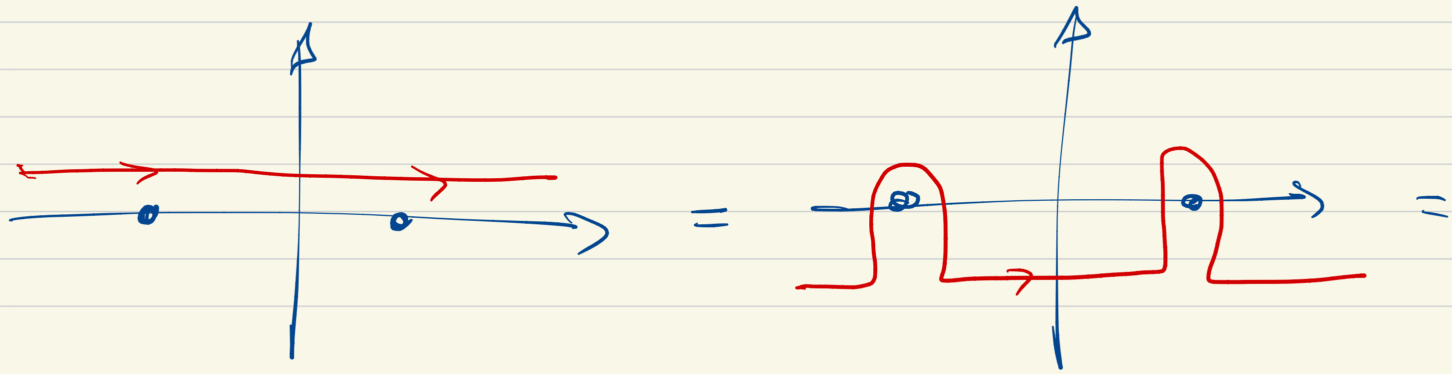


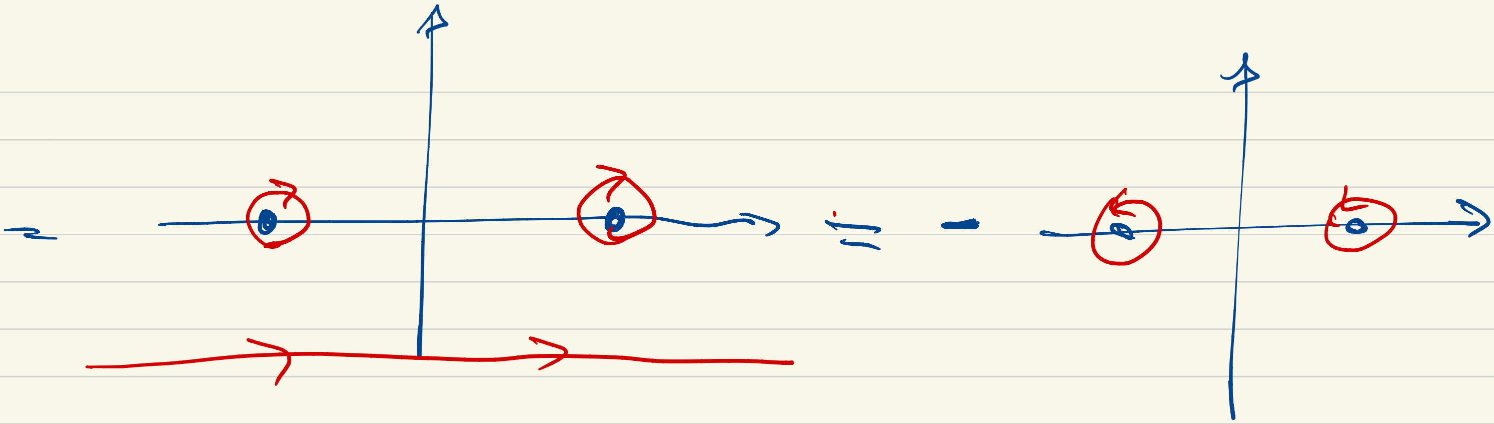
$t - t' < 0$

$i\omega \to -\infty$

$e^{-i\omega(t-t')} \to 0$

Now let us consider $t - t' > 0$. Our goal is to deform the contour to simplify the integral.





$$\int \frac{d^3k\,d\omega}{(2\pi)^4}\, e^{ik(x-x')-i\omega(t-t')}\, \frac{c^2}{-(\omega-ck)(\omega+ck)} =$$

$\vec{r} = \vec{x} - \vec{x}'$

$\omega = ck$ $-$ $\omega = -ck$ $+$

$$= -2\pi i \cdot \frac{c^2}{(2\pi)^4} \int d^3k\, \frac{e^{i\vec{k}(\vec{x}-\vec{x}')}}{2ck} \left(e^{ick(t-t')} - e^{-ick(t-t')} \right)$$

$$2\pi \int_0^\pi d\theta \sin\theta\, e^{ikr\cos\theta} = \frac{2\pi}{ikr}\left(e^{ikr} - e^{-ikr}\right)$$

$$-\frac{2\pi^3 \cdot c}{r} \cdot \int_0^\infty \frac{dk}{(2\pi)^4} \Big[e^{ik(r+c(t-t'))} +$$

$$+ e^{-ik(r+c(t-t'))} - e^{ik(r-c(t-t'))} - e^{-ik(r-c(t-t'))} \Big]$$

$$\int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ik\alpha} = \delta(\alpha).$$

$t > t' \Rightarrow$ only one term survives

$$G_R = \frac{\delta(t - t' - \frac{r}{c})}{4\pi r} \qquad G_A = \frac{\delta(t - t' + \frac{r}{c})}{4\pi r}$$

$$\varphi(x) = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho(x', t - \frac{|x-x'|}{c})}{|x-x'|} d^3x'$$

$$\vec{A}(x) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(x', t - \frac{|x-x'|}{c})}{|x-x'|} d^3x'$$

$$\left[-\frac{\partial_t^2}{c^2} + \frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} \right] \frac{\delta\left(t - t' - \frac{r}{c}\right)}{r} =$$

$$= \delta(t - t') \cdot \delta^3(\vec{r})$$